Int. J. Solids Structures Vol. 30, No. 16, pp. 2277–2285, 1993
Printed in Great Britain

0020–7683/93 $6.00+ .00

FINITE DEFORMATIONS OF POLAR ELASTIC MEDIA

PAWEŁ H. DŁUŻEWSKI
Institute of Fundamental Technological Research, Swietokrzyska 21, 00-049 Warsaw, Poland

(*Received* 27 *January* 1992; *in revised form* 25 *February* 1993)

Abstract—In many papers on oriented continua some orthonormal angular coordinates were proposed. With respect to the curvature of the orientation space, it is obvious that such coordinates could not be applied in practice. Therefore, instead of these coordinates a tensor field of rotations had to be used to define the wryness tensor.

In this paper the curvilinear coordinates in orientation space are considered. The Euler angles are an example of such coordinates. The inertia conservation law is replaced here by a constitutive relation. From the physical point of view this relation is more general and seems to be better justified than the mentioned law. Within the framework of the polar continuum theory a micromorphic structure is discussed. Some remarks on the principle of a material frame-indifference are also presented.

NOMENCLATURE

$\mathscr{g}_{\alpha\beta}$	angular metric tensor in the current configuration
g_{kl}	metric tensor in the current configuration
$\mathscr{G}_{\Theta\Phi}$	angular metric tensor in the reference configuration
G_{KL}	metric tensor in the reference configuration
$g^K_k, g^k_\alpha, g^\alpha_\Theta$	shifters
$e_{\alpha kl}, E_{\Theta KL}$	alternating tensors
ρ	mass density
j^{kl}	tensor of inertia
i^{kl}	Euler tensor of inertia
v^k	velocity vector
ω^α	angular velocity vector
f^k	external force density
l^α	external angular force density
k^α	angular momentum density
t^{kl}	stress tensor
$m^{\alpha k}$	couple stress tensor
h	heat source density
q^k	heat vector
η	entropy density
ε	internal energy density
ψ	free energy density
T	temperature
$\mathbf{F}$	deformation gradient
$\mathscr{F}$	angular deformation gradient
$\mathfrak{E}$	strain tensor
$\mathfrak{D}$	angular strain tensor
$\mathfrak{T}$	stress tensor
$\mathfrak{M}$	couple stress tensor
$\mathbf{\Gamma}$	wryness tensor
$\mathbf{Q}$	rotation tensor
$\mathbf{R}$	orthogonal part of the deformation gradient
$\mathbf{U}$	symmetric part of the deformation gradient
$\mathfrak{U}$	stretch tensor.

1. INTRODUCTION

In 1909 E. and F. Cosserat published the monograph on the continuum mechanics of oriented bodies. Later, the theory was developed by Toupin (1964), Eringen and Kafadar (1976), Nowacki (1971), and Ericksen (1976) among many others. In consequence, many various constitutive descriptions of polar deformations have been proposed. To distinguish between the models they were called: polar bodies, Cosserat media, microstructural, micromorphic, microstretch media and the like.

The concept developed here goes back to the idea of Kafadar and Eringen (1971). In the next section a description of the position and orientation of a polar particle in spatial and angular coordinates is discussed. The Euler angles are used as an example of coordinates in the curvature space of orientations. The third section deals with field equations derived from the balance laws. The fourth section is devoted to the constitutive description of the finite deformations of polar media. Next, a micromorphic structure is considered. Finally, the problem of the reference frame objectivity of the constitutive equations for the elastic polar medium is discussed.

2. KINEMATICS

Let $\mathbf{x}$ be a vector describing a position of a particle κ in a three-dimensional Euclidean space $\mathbb{R}$, where (x^1, x^2, x^3) denote components of the position in a curvilinear coordinate system $\{x^k\}$. Let $\mathbf{e}_k$ denote the k-base vector tangent to the k-coordinate line at the point $\mathbf{x}$. Evidently, $\mathbf{x} \neq x^k\mathbf{e}_k$; however, for any vector $\mathbf{v}$ applied at κ we have $\mathbf{v} = v^k\mathbf{e}_k$. Similarly, let some coordinates, e.g. the Euler angles (ϕ^1, ϕ^2, ϕ^3), describe the orientation $\boldsymbol{\phi}$ of κ in the standard orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ of the Euclidean space $\mathbb{R}$. According to Fig. 1, the components (X^K) denote the reference position in terms of the curvilinear coordinate system $\{X^K\}$ in $\mathbb{R}$. Similarly, the components (Φ^Θ) determine the reference orientation of the particle in terms of the angular coordinate system $\{\Phi^\Theta\}$ in the orientation space.

It can be shown that for any geometric object (e.g. a crystal lattice) we can determine the object orientation space $\mathscr{R}$ being the constant curvature space [see e.g. Dłużewski (1991a)]. For any infinitesimal change of the object orientation through an angle $d\alpha$ we find

$$(d\alpha)^2 = g_{\beta\gamma}\, d\phi^\beta\, d\phi^\gamma. \tag{1}$$

Each rotation around an immobile axis marks a geodesic line in this space. The angle between two different orientations of the object is the Riemannian distance. With respect to the curvature of the orientation space it is impossible to introduce any set of orthogonal

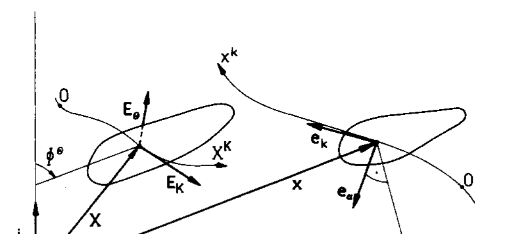

Fig. 1. Bases and coordinate lines.

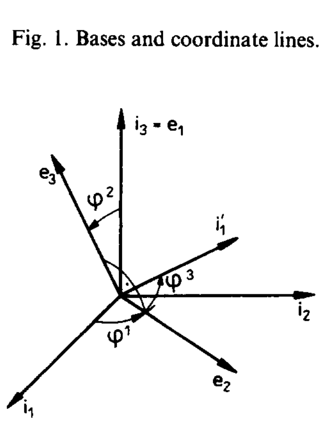

Fig. 2. Euler angles ϕ^1, ϕ^2, ϕ^3 understood as the angular coordinates; $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$—the angular base vectors.

coordinates in $\mathscr{R}$ for which the finite angular distance θ between two different orientations could satisfy $\theta = \sqrt{\phi^{\alpha}\phi_{\alpha}}$. Such coordinates were proposed in many papers [cf. Eringen and Kafadar (1976)]. Obviously, they could not be used in practice.

In the case of the correctly defined coordinates, as curvilinear ones on the curvature space, many tensorial quantities may be considered in terms of angular coordinates. For example, the angular velocity vector $\boldsymbol{\omega}$, can be considered in terms of

$$\omega^{\alpha} \equiv \dot{\phi}^{\alpha} = \frac{\partial \phi^{\alpha}}{\partial t}. \tag{2}$$

The same vector can also be considered in the generally known curvilinear terms ω^k on $\mathbb{R}$. They are correlated by

$$\boldsymbol{\omega} = \omega^{\alpha}\mathbf{e}_{\alpha} = \omega^{k}\mathbf{e}_{k}, \tag{3}$$

where the angular base vectors $\mathbf{e}_{\alpha}$ are parallel to the axes of the instantaneous rotations $\mathrm{d}\phi^{\alpha}$, respectively (see Fig. 2) [cf. Skalmierski (1991)].

The angular velocity tensor is defined here with the help of the angular velocity vector

$$\omega_{kl} \equiv -e_{\alpha kl}\omega^{\alpha}, \tag{4}$$

where $e_{\alpha kl}$ is the two-point representation of the alternating tensor (see Appendix).

The motion of a polar continuum in relation to the reference configuration will be described by two maps $\mathbf{x}(\mathbf{X}, t)$ and $\boldsymbol{\phi}(\mathbf{X}, t)$. We assume that for the reference configuration the mapping $\boldsymbol{\Phi}(\mathbf{X})$ is given. The gradient of the displacement deformation is defined as

$$F^{k}_{K} \equiv x^{k}_{,K} = \frac{\partial x^{k}}{\partial X^{K}}, \tag{5}$$

[cf. (19.1) in Ericksen (1960)]. Using the angular coordinates we can also define the angular deformation gradient:

$$\mathscr{F}^{\alpha}_{K} \equiv \phi^{\alpha}_{,K} = \frac{\partial \phi^{\alpha}}{\partial X^{K}}. \tag{6}$$

In many cases the rotation of a microstructure cannot be determined on the basis of so-called polar decomposition of the deformation gradient $\mathbf{F} = \mathbf{RU}$, e.g. the crystal lattice rotation in deformed grains of polycrystal. In such a case we may use the following "polar" decomposition of deformation gradients in the rotation of microstructure and its deformation, i.e.

$$\mathbf{F} = \mathbf{Q}\mathfrak{C},$$
$$\mathscr{F} = \mathbf{Q}\boldsymbol{\Gamma}. \tag{7}$$

This corresponds to the following definitions:

$$\mathfrak{C} \equiv \mathbf{Q}^{\mathrm{T}}\mathbf{F},$$
$$\boldsymbol{\Gamma} \equiv \mathbf{Q}^{\mathrm{T}}\mathscr{F}. \tag{8}$$

The corotational gradients $\mathfrak{C}$ and $\boldsymbol{\Gamma}$ are often called the Cosserat deformation tensor and the wryness tensor, respectively. Let us also assume that

$$\mathfrak{C} = \mathfrak{C}_0 + \mathfrak{E},$$
$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_0 + \mathfrak{D}, \tag{9}$$

where

$$\Gamma_0{}^{\Theta}{}_K \equiv \Phi^{\Theta}_{,K} = \frac{\partial \Phi^{\Theta}}{\partial X^K} \quad \text{and} \quad \mathfrak{C}_0{}^K{}_L \equiv X^K_{,L} = \delta^K_L.$$

It corresponds to the important definitions

$$\begin{aligned} \mathfrak{E} &\equiv \mathbf{Q}^{\mathrm{T}}\mathbf{F} - \mathbf{F}_0, \\ \mathfrak{D} &\equiv \mathbf{Q}^{\mathrm{T}}\mathscr{F} - \mathscr{F}_0, \end{aligned} \tag{10}$$

which can be written in the form

$$\begin{aligned} \mathfrak{E}^K{}_L &\equiv \mathbf{Q}_k{}^K x^k_{,L} - X^K_{,L}, \\ \mathfrak{D}^{\Theta}{}_L &\equiv Q_\alpha{}^{\Theta}\phi^\alpha_{,L} - \Phi^{\Theta}_{,L}, \end{aligned} \tag{11}$$

where $Q_\alpha{}^{\Theta} = Q_k{}^K g^k_\alpha g^{\Theta}_K$ (see also Appendix).

3. BALANCE LAWS

For a polar medium the balance laws have been discussed in many papers [e.g. Nowacki (1971) and Kafadar and Eringen (1971)]. The mass balance, momentum balance, moment of momentum balance and energy balance are stated respectively by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_v \rho \,\mathrm{d}v = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_v \rho\mathbf{v}\,\mathrm{d}v = \int_s \mathbf{t}_{(n)}\,\mathrm{d}s + \int_v \rho\mathbf{f}\,\mathrm{d}v,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_v (\mathbf{x}\times\rho\mathbf{v}+\rho\mathbf{k})\,\mathrm{d}v = \int_s (\mathbf{x}\times\mathbf{t}_{(n)}+\mathbf{m}_{(n)})\,\mathrm{d}s + \int_v (\mathbf{x}\times\rho\mathbf{f}+\rho\mathbf{l})\,\mathrm{d}v,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_v (\rho\varepsilon + \tfrac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} + \tfrac{1}{2}\rho\mathbf{k}\cdot\boldsymbol{\omega})\,\mathrm{d}v = \int_s (\mathbf{t}_{(n)}\cdot\mathbf{v} + \mathbf{m}_{(n)}\cdot\boldsymbol{\omega})\,\mathrm{d}s + \int_v (\rho\mathbf{f}\cdot\mathbf{v} + \rho\mathbf{l}\cdot\boldsymbol{\omega})\,\mathrm{d}v$$

$$-\int_s \mathbf{q}\cdot\mathrm{d}\mathbf{s} + \int_v \rho h\,\mathrm{d}v, \tag{12}$$

where $\mathbf{t}_{(n)} = t^{kl}n_l\mathbf{e}_k$, $\mathbf{m}_{(n)} = m^{\alpha k}n_k\mathbf{e}_\alpha$, $\mathbf{n}$ denotes the unit normal to the surface s bounding the polar body region v, and $\times$ denotes the vector product. The above laws give the following field equations:†

$$\begin{aligned} \rho_{,t} + (\rho v^k)_{,k} &= 0, \\ t^{kl}{}_{,l} + \rho f^k &= \rho\dot{v}^k, \\ m^{\alpha k}{}_{,k} + 2t^\alpha + \rho l^\alpha &= \rho\dot{k}^\alpha, \\ \rho\dot{\varepsilon} - \tfrac{1}{2}\rho\omega_\alpha \dot{j}^{\alpha\beta}\omega_\beta &= t^{kl}v_{k,l} - 2t^\alpha\omega_\alpha + m^{\alpha k}\omega_{\alpha,k} - q^k_{,k} + \rho h, \end{aligned} \tag{13}$$

where $t^\alpha \equiv -\frac{1}{2}\,\mathrm{e}^{\alpha mn}\,t_{mn}$ and $k^\alpha \equiv j^{\alpha k}\omega_k$.

† Except for the differentiation over time ($\rho_{,t} \equiv \partial\rho/\partial t$) a comma denotes the covariant differentiation, e.g.

$$\omega^\alpha_{,k} = \frac{\partial\omega^\alpha}{\partial x^k} + \frac{\omega^\beta\,\partial\mathbf{e}_\beta}{\partial x^k}\cdot\mathbf{e}^\alpha, \quad \text{where} \quad \Gamma_{k\beta}{}^\alpha \stackrel{\mathrm{def}}{=} \frac{\partial\mathbf{e}_\beta}{\partial x^k}\,\mathbf{e}^\alpha.$$

Note that $\Gamma_{k\beta}{}^\alpha \neq \Gamma_{\beta k}{}^\alpha$. It means that the connection coefficients are not generated by metric tensors, this problem is often discussed in the non-Riemannian geometries.

In the case of rigid body mechanics the moment of inertia is defined by

$$\hat{j}_{kl} = \int_v \rho(\delta_{kl} z_m z_m - z_k z_k)\,\mathrm{d}v = \delta_{kl}\hat{i}_{mm} - \hat{i}_{kl}, \tag{14}$$

where

$$\hat{i}_{kl} = \int_v \rho z_k z_l\,\mathrm{d}v, \tag{15}$$

z_k denotes the k-coordinate in the standard orthonormal coordinate system in $\mathbb{R}$. By analogy to the rigid body mechanics we assume

$$j_{\alpha\beta} = g_{\alpha\beta} i^{\gamma}{}_{\gamma} - i_{\alpha\beta}. \tag{16}$$

After Eringen in many papers an additional balance law for conservation of inertia is assumed [see e.g. Eringen and Kafadar (1976)]. In my opinion we have no reason to assume that the inertia of particles cannot change when the whole continuum is deformed. Therefore, we assume here that the inertia of polar particles may change together with the deformation of the continuum.

Entropy inequality

The entropy inequality is stated for a polar body to be

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_v \rho\eta\,\mathrm{d}v \geqslant -\int_s \frac{\mathbf{q}\cdot\mathrm{d}\mathbf{s}}{T} + \int_v \frac{\rho h}{T}\,\mathrm{d}v. \tag{17}$$

This leads to the field equation

$$\rho\dot{\eta} + \left(\frac{q^k}{T}\right)_{,k} - \frac{\rho h}{T} \geqslant 0. \tag{18}$$

Using $(13)_4$, the last inequality can be expressed by

$$-\rho\dot{\psi} - \rho\eta\dot{T} + \tfrac{1}{2}\rho\omega^{\alpha}\dot{j}_{\alpha\beta}\omega^{\beta} + t^{kl}v_{k,l} - 2t^{\alpha}\omega_{\alpha} + m^{\alpha k}\omega_{\alpha,k} - \frac{q^k}{T}T_{,k} \geqslant 0, \tag{19}$$

where $\psi = \varepsilon - \eta T$.

4. POLAR ELASTICITY

Let us assume that the free energy of a polar particle depends on the particle strain and temperature

$$\psi = \psi(\mathbf{X}, \mathfrak{E}, \mathfrak{D}, T). \tag{20}$$

Assume also that

$$\mathbf{j} = \mathbf{Q}\mathbf{J}(\mathbf{X}, \mathfrak{E}, \mathfrak{D}, T)\mathbf{Q}^{\mathrm{T}}. \tag{21}$$

In other words, we assume that the moment of inertia may change together with the polar particle deformation and the temperature. In our case

$$\dot{\psi} = \frac{\partial \psi}{\partial \mathfrak{E}} \dot{\mathfrak{E}} + \frac{\partial \psi}{\partial \mathfrak{D}} \dot{\mathfrak{D}} + \frac{\partial \psi}{\partial T} \dot{T}, \tag{22}$$

and

$$\dot{\mathbf{j}} = \dot{\mathbf{Q}}\mathbf{J}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\left(\frac{\partial \mathbf{J}}{\partial \mathfrak{E}} \dot{\mathfrak{E}} + \frac{\partial \mathbf{J}}{\partial \mathfrak{D}} \dot{\mathfrak{D}} + \frac{\partial \mathbf{J}}{\partial T} \dot{T}\right)\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{J}\dot{\mathbf{Q}}^{\mathrm{T}}, \tag{23}$$

where

$$\begin{aligned}
\dot{\mathfrak{E}}^{K}{}_{L} &= -\omega^{\alpha} e_{\alpha k}{}^{l} Q_{l}{}^{K} x^{k}_{,L} + Q_{k}{}^{K} v^{k}_{,l} x^{l}_{,L}, \\
\dot{\mathfrak{D}}^{\Theta}{}_{L} &= \omega^{\alpha}_{,k} x^{k}_{,L} Q_{\alpha}{}^{\Theta}, \\
\dot{Q}^{\alpha}{}_{\Theta} &= -\omega^{\gamma} e_{\gamma}{}^{\alpha}{}_{\beta} Q^{\beta}{}_{\Theta}.
\end{aligned} \tag{24}$$

Substituting (22) and (23) into (19) we find

$$\begin{aligned}
&-\rho\left(\frac{\partial \psi}{\partial T} + \eta - \tfrac{1}{2}\omega^{\alpha} Q_{\alpha}{}^{\Lambda} \frac{\partial J_{\Lambda\Phi}}{\partial T} Q_{\beta}{}^{\Phi}\omega^{\beta}\right)\dot{T} \\
&\quad + v^{k}_{,l}\left(t_{k}{}^{l} - \rho \frac{\partial \psi}{\partial \mathfrak{E}^{K}{}_{L}} Q_{k}{}^{K} x^{l}_{,L} + \tfrac{1}{2}\rho\omega^{\gamma} Q_{\gamma}{}^{\Lambda} \frac{\partial J_{\Lambda\Phi}}{\partial \mathfrak{E}^{K}{}_{L}} Q_{k}{}^{K} x^{l}_{,L} Q_{\beta}{}^{\Phi}\omega^{\beta}\right) \\
&\quad - \omega^{\alpha}\left(2t_{\alpha} - \rho \frac{\partial \psi}{\partial \mathfrak{E}^{K}{}_{L}} e_{\alpha k}{}^{l} Q_{l}{}^{K} x^{k}_{,L} + \tfrac{1}{2}\rho\omega^{\gamma} Q_{\gamma}{}^{\Lambda} \frac{\partial J_{\Lambda\Phi}}{\partial \mathfrak{E}^{K}{}_{L}} e_{\alpha k}{}^{l} Q_{l}{}^{K} x^{k}_{,L} Q_{\beta}{}^{\Phi}\omega^{\beta}\right) \\
&\quad + \omega^{\alpha}_{,k}\left(m_{\alpha}{}^{k} - \rho \frac{\partial \psi}{\partial \mathfrak{D}^{\Theta}{}_{L}} Q_{\alpha}{}^{\Theta} x^{k}_{,L} + \tfrac{1}{2}\rho\omega^{\gamma} Q_{\gamma}{}^{\Lambda} \frac{\partial J_{\Lambda\Phi}}{\partial \mathfrak{D}^{\Theta}{}_{L}} Q_{\alpha}{}^{\Theta} x^{k}_{,L} Q_{\beta}{}^{\Phi}\omega^{\beta}\right) - \frac{q^{k}}{T} T_{,k} \geqslant 0.
\end{aligned} \tag{25}$$

Assuming that the constitutive equation for heat flux, e.g.

$$\mathbf{q} = \mathbf{Q}\hat{\mathbf{q}}(\mathbf{X}, \mathfrak{E}, \mathfrak{D}, T, \mathbf{Q}^{\mathrm{T}}\nabla T), \tag{26}$$

must satisfy $(q^{k}/T)T_{,k} \leqslant 0$ we conclude that

$$\begin{aligned}
\eta &= -\frac{\partial \psi}{\partial T} + \tfrac{1}{2}\omega^{\alpha} Q_{\alpha}{}^{\Lambda} \frac{\partial J_{\Lambda\Phi}}{\partial T} Q_{\beta}{}^{\Phi}\omega^{\beta}, \\
t_{k}{}^{l} &= \rho Q_{k}{}^{K}\left(\frac{\partial \psi}{\partial \mathfrak{E}^{K}{}_{L}} - \tfrac{1}{2}\omega^{\gamma} Q_{\gamma}{}^{\Lambda} \frac{\partial J_{\Lambda\Phi}}{\partial \mathfrak{E}^{K}{}_{L}} Q_{\beta}{}^{\Phi}\omega^{\beta}\right) x^{l}_{,L}, \\
m_{\alpha}{}^{k} &= \rho Q_{\alpha}{}^{\Theta}\left(\frac{\partial \psi}{\partial \mathfrak{D}^{\Theta}{}_{L}} - \tfrac{1}{2}\omega^{\gamma} Q_{\gamma}{}^{\Lambda} \frac{\partial J_{\Lambda\Phi}}{\partial \mathfrak{D}^{\Theta}{}_{L}} Q_{\beta}{}^{\Phi}\omega^{\beta}\right) x^{k}_{,L}.
\end{aligned} \tag{27}$$

The deformation measures $\mathfrak{E}$ and $\mathfrak{D}$ are conjugated with the following stress measures:

$$\begin{aligned}
\mathfrak{T} &= \rho_0 \frac{\partial \psi}{\partial \mathfrak{E}} - \tfrac{1}{2}\rho_0 \omega \mathbf{Q} \frac{\partial \mathbf{J}}{\partial \mathfrak{E}} \mathbf{Q}^{\mathrm{T}}\omega, \\
\mathfrak{M} &= \rho_0 \frac{\partial \psi}{\partial \mathfrak{D}} - \tfrac{1}{2}\rho_0 \omega \mathbf{Q} \frac{\partial \mathbf{J}}{\partial \mathfrak{D}} \mathbf{Q}^{\mathrm{T}}\omega,
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
\mathfrak{T} &\equiv \frac{\rho_0}{\rho} \mathbf{Q}^{\mathrm{T}}\mathbf{t}(\mathbf{F}^{-1})^{\mathrm{T}}, \\
\mathfrak{M} &\equiv \frac{\rho_0}{\rho} \mathbf{Q}^{\mathrm{T}}\mathbf{m}(\mathbf{F}^{-1})^{\mathrm{T}}.
\end{aligned} \tag{29}$$

Thus, the entropy inequality can be rewritten as

$$\rho_0\,\mathrm{d}\psi - \tfrac{1}{2}\rho_0\omega\mathbf{Q}\,\mathrm{d}\mathbf{J}\,\mathbf{Q}^{\mathrm{T}}\omega \leqslant \mathfrak{T}\,\mathrm{d}\mathfrak{E} + \mathfrak{M}\,\mathrm{d}\mathfrak{D} + \eta_0\,\mathrm{d}T, \tag{30}$$

where $\eta_0 = (\rho_0/\rho)\eta$.

Using the polar continuum theory the stretch of a polar particle can be described by the following constitutive equation,

$$\mathfrak{U} = \mathfrak{U}(\mathbf{X}, \mathfrak{E}, \mathfrak{D}, T). \tag{31}$$

In such a case the Euler inertia of the particle is determined by

$$\mathbf{i} = \mathbf{Q}\mathfrak{U}\mathbf{I}\mathfrak{U}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}, \tag{32}$$

where $\mathbf{I}$ is the inertia in the reference configuration, i.e. $\mathbf{I} = \mathbf{I}(\mathbf{X})$. In such a case the general constitutive relation (21) can be replaced by more precise relations: (16), (31) and (32).

Comparing the present approach with the Eringen polar and micromorphic theories the reader may find many differences. First of all in Eringen's approach the inertia of particles cannot change, i.e. the tensor of inertia rotates together with the particle. In other words the microstretch of a polar particle does not induce any change of its inertia. In the present approach the inertia may change together with the particle deformation [see e.g. eqns (31) and (32)].

5. MICROMORPHIC STRUCTURE

Let us assume that a given material is composed of various microelements. By an orientation of polar particles we may take an orientation of an arbitrary chosen microelement within the polar particle, e.g. the orientation of the triangle in Fig. 3.

Using the polar continuum theory we may predict deformations of other microelements of the particle. Let the local rotation of a second microelement be governed by the relation

$$\mathfrak{R} = \mathfrak{R}(\mathbf{X}, \mathfrak{E}, \mathfrak{D}, T). \tag{33}$$

For example, let $\mathfrak{R}$ denote a local rotation of the square in relation to the orientation basis identified with the triangle in Fig. 3. In a similar manner the stretch of the square can be described by

$$\mathfrak{U}^* = \mathfrak{U}^*(\mathbf{X}, \mathfrak{E}, \mathfrak{D}, T). \tag{34}$$

In such a case the corotational gradient of the square deformation observed from the position of orientation basis is determined by $\mathfrak{F} = \mathfrak{R}\mathfrak{U}^*$.

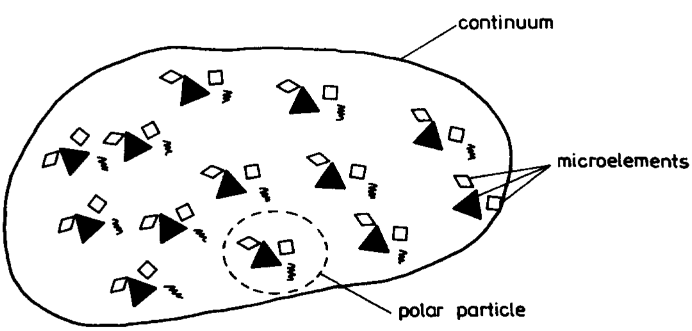

Fig. 3. Example of micromorphic structure.

Table 1. Dependence of strain measures on the orientation basis assumed for polar particles

	Orientation basis	
	Identified with the first microelement	Identified with the second microelement
Polar particle rotation	$\mathbf{Q}$	$\mathbf{Q}^* = \mathbf{Q}\mathfrak{R}$
Displacement deformation of polar particle	$\mathfrak{E}$	$\mathfrak{E}^* = \mathfrak{R}^T\mathfrak{E}$
Angular deformation of polar particle	$\mathfrak{D}$	$\mathfrak{D}^* = \mathfrak{R}^T\mathfrak{D}$
Deformation of the first microelement	$\mathfrak{U}$	$\mathfrak{R}^T\mathfrak{U}$
Deformation of the second microelement	$\mathfrak{R}\mathfrak{U}^*$	$\mathfrak{U}^*$

Now, instead of the triangle let us take the square as the orientation basis. From the viewpoint of the new basis the polar particle strains may reach new values $\mathfrak{E}^*$ and $\mathfrak{D}^*$ (see Table 1). Consequently, the constitutive equations (20) and (21) transform into

$$\psi = \psi^*(\mathbf{X}, \mathfrak{E}^*, \mathfrak{D}^*, T),$$
$$\mathbf{J} = \mathbf{J}^*(\mathbf{X}, \mathfrak{E}^*, \mathfrak{D}^*, T). \tag{35}$$

Hence, the considered equations depend on what we mean by the orientation basis for a polar particle. On the other hand, their general form is independent of the orientation basis choice.

6. DISCUSSION

It has been shown that the thermodynamically admissible constitutive equations for a polar elastic medium are given by eqns (20), (21) and (26). The price which we have to pay for the rejection of the conservation law for inertia is a more advanced form of the constitutive equations (27).

Equations (27) may seem to be non-objective and there may arise some questions. Let us consider the axiom of the material frame-indifference. According to (27) the stresses determined by one observer may reach different values in comparison with the stresses calculated by another observer moving with the translatory and rotary motion in relation to the first one. This is nonsense because the axiom cannot be applied indirectly to (27). In the case of a system of rigid bodies it is generally known that if the first observer is in an inertial system then the second observer is also in an inertial system only if the second one is in the steady translatory motion in relation to the first one. Otherwise, the second observer is in a non-inertial system, e.g. for the steady translatory and rotary motion. In such a case the local balance laws for polar continuum take a form different from $(13)_{2,3,4}$ and (19), see e.g. the Coriolis acceleration. Because of the limited size of the paper the present considerations concern the polar body behaviour in the inertial frame. However, replacing (19) by the equations proper for a relative motion it can also be proved that the polar elastic media may be defined by (20), (21) and (26).

Acknowledgement—This research was supported by the Polish Committee of Scientific Research (KBN) under Grant No. 3 3334 91 2.

REFERENCES

Cosserat, E. and Cosserat, F. (1909). *Theorie des Corps Deformable.* Hermann, Paris.
Dłużewski, P. H. (1991a). Crystal orientation spaces and remarks on the modelling of polycrystal anisotropy. *J. Mech. Phys. Solids* **39,** 651–661.
Dłużewski, P. H. (1991b). Finite deformations of polar media in angular coordinates. *Arch. Mech.* **43,** 783–793.
Ericksen, J. L. (1960). Tensor fields. In *Encyclopedia of Physics* (Edited by C. Truesdell and R. Toupin), Vol. III, Pt 1. Springer, Berlin.

Ericksen, J. L. (1976). Equilibrium theory of liquid crystals. In *Advances in Liquid Crystals* (Edited by G. H. Brown), Vol. 2.
Eringen, A. C. and Kafadar, C. B. (1976). Polar field theories. In *Continuum Physics* (Edited by A. C. Eringen), Vol. 4, pp. 1–73. Academic Press, New York.
Kafadar, C. B. and Eringen, A. C. (1971). Micropolar media—I. The classical theory. *Int. J. Engng Sci.* **9**, 271–305.
Nowacki, W. (1971). *Teoria niesymetrycznej sprężystości.* PWN, Warszawa.
Skalmierski, B. (1991). Mechanics. PWN, Warsaw and Elsevier, Amsterdam.
Toupin, R. A. (1964). Theory of elasticity with couple stress. *Arch. Rat. Mech. Analy.* **17**, 85.

APPENDIX

For a polar medium many vectors can be shifted in the position space $\mathbb{R}$ as well as in the orientation space $\mathscr{R}$. The following shifters are defined on $\mathbb{R}$:

$$\begin{aligned} g^k_K &\equiv \mathbf{e}^k \cdot \mathbf{E}_K, \\ g^K_k &\equiv \mathbf{E}^K \cdot \mathbf{e}_k. \end{aligned} \tag{A1}$$

We also define the mixed shifters:

$$\begin{aligned} g^\alpha_k &\equiv \mathbf{e}^\alpha \cdot \mathbf{e}_k, \\ g^k_\alpha &\equiv \mathbf{e}^k \cdot \mathbf{e}_\alpha, \\ g^K_\Theta &\equiv \mathbf{E}^K \cdot \mathbf{E}_\Theta, \\ g^\Theta_K &\equiv \mathbf{E}^\Theta \cdot \mathbf{E}_K. \end{aligned} \tag{A2}$$

Besides the following angular shifters can be defined

$$\begin{aligned} g^\alpha_\Theta &\equiv \mathbf{e}^\alpha \cdot \mathbf{E}_\Theta, \\ g^\Theta_\alpha &\equiv \mathbf{E}^\Theta \cdot \mathbf{e}_\alpha. \end{aligned} \tag{A3}$$

It is easy to prove that $g^k_K = g^k_\alpha g^\alpha_\Theta g^\Theta_K$.

Alternating tensors are defined at the points x and X of $\mathbb{R}$ as follows:

$$e_{klm} \equiv \pm \varepsilon_{klm}\sqrt{\det(\mathbf{g})}, \quad E_{KLM} \equiv \pm \varepsilon_{KLM}\sqrt{\det(\mathbf{G})}, \tag{A4}$$

where ε_{klm} is the permutation symbol and the sign $\pm$ is positive for a right-handed coordinate system and negative for a left-handed one. Similarly, at the points ϕ and Φ of $\mathscr{R}$ we may define, respectively,

$$e_{\alpha\beta\gamma} \equiv +\varepsilon_{\alpha\beta\gamma}\sqrt{\det(\mathscr{g})}, \quad \mathscr{E}_{\Theta\Phi\Lambda} \equiv \varepsilon_{\Theta\Phi\Lambda}\sqrt{\det(\mathscr{G})}. \tag{A5}$$

Note that the shifters introduced above lead to the following important relationships:

$$g_{kl} = g^K_k g^L_l G_{KL} = g^\Theta_k g^\Phi_l \mathscr{G}_{\Theta\Phi} = g^\alpha_k g^\beta_l \mathscr{g}_{\alpha\beta} \tag{A6}$$

and

$$e_{klm} = g^K_k g^L_l g^M_m E_{KLM} = g^\Theta_k g^\Phi_l g^\Lambda_m \mathscr{E}_{\Theta\Phi\Lambda} = g^\alpha_k g^\beta_l g^\gamma_m e_{\alpha\beta\gamma}. \tag{A7}$$